Zen and the Art of ∞-Categories

or

How I learnt to stop worrying and love ∞-category theory*

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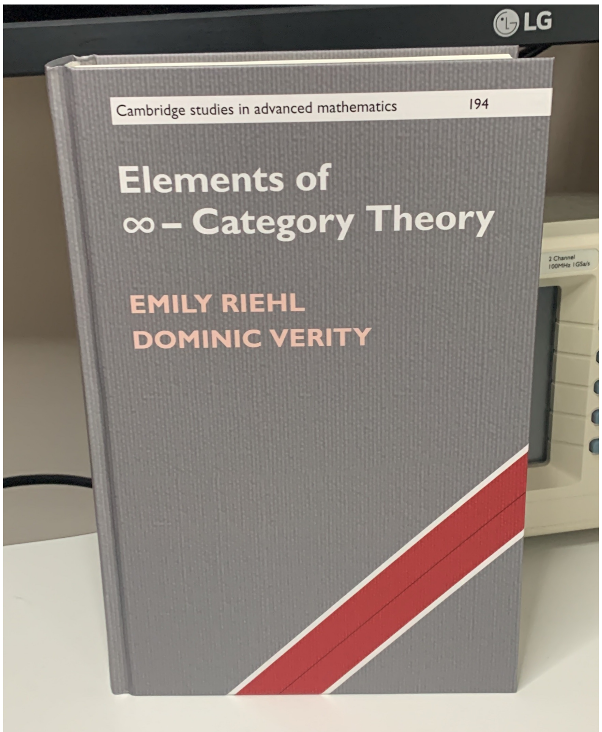
I acknowledge the traditional owners of the land on which this talk takes place, the Wattamattagal clan of the Darug nation, whose cultures and customs have nurtured this land since the Dreamtime.

I pay my respects to Elders past, present and emerging.

THE OBLIGATORY PLUG



EVERYTHING YOU WANTED TO KNOW ABOUT MODEL AGNOSTIC ∞-CATEGORY THEORY, BUT WERE TOO AFRAID TO ASK!!

DOM VERITY (MQ)

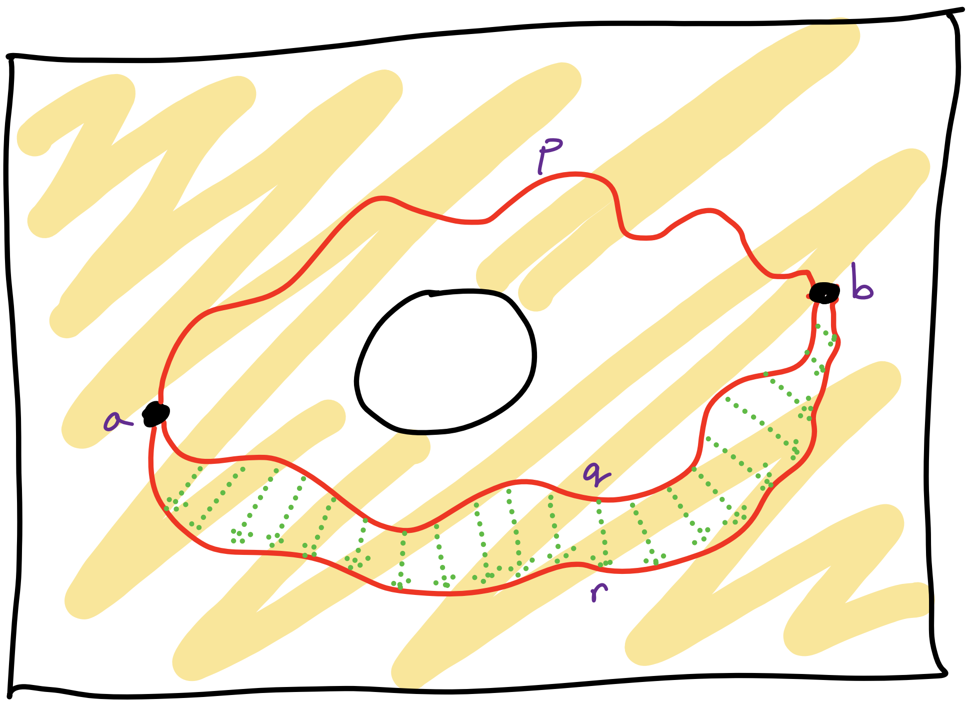
EMILY RIEHL (JHU)

PARTNERS IN ∞-CRIME

BUILDING UP TO ∞-CATS

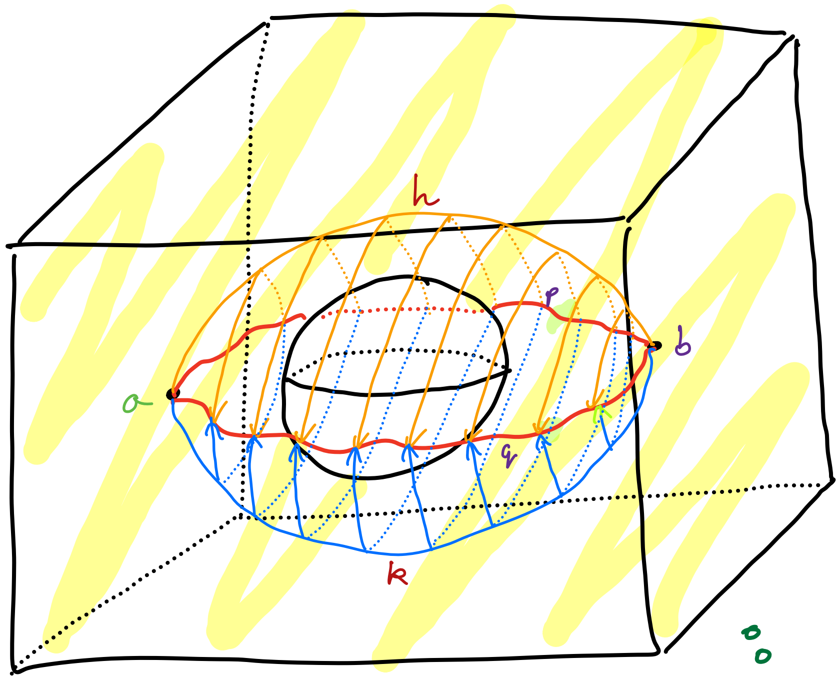
THE FUNDAMENTAL ∞-GROUPOID

A FAMILIAR STORY



FUNDAMENTAL GROUPOID $\pi_1(X)$

- OBJECTS POINTS IN THE SPACE
- ARROWS BASED HOMOTOPY CLASSES OF CONTINUOUS PATHS.
- COMPOSITION CONCATENATION OF PATHS.
- INVERSES GIVEN BY TRAVERSING PATHS IN THE REVERSE DIRECTION



HERE THE PATHS p AND q ARE HOMOTOPIC VIA TWO DISTINCT HOMOTOPIES h AND k WHICH CIRCUMNAVIGATE THE CENTRAL HOLE.

FUNDAMENTAL ∞-GROUPOID $\Pi(X)$

- OBJECTS POINTS IN THE SPACE
- ARROWS CONTINUOUS PATHS BETW POINTS
- HOMOTOPIES BETWEEN PATHS
- 2-HOMOTOPIES BETWEEN HOMOTOPIES
- 3-HOMOTOPIES BETWEEN 2-HOMOTOPIES

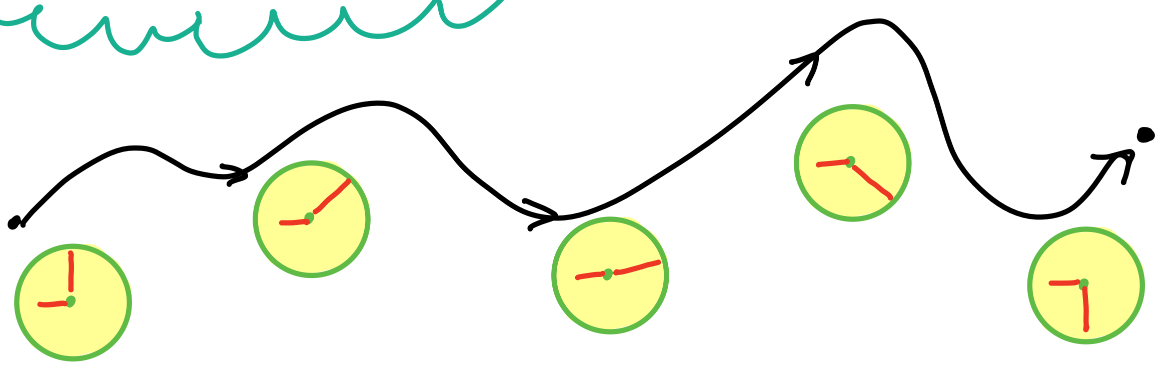
⋮

DIRECTED SPACES
&
THE FUNDAMENTAL $(\infty,1)$-CATEGORY

IN PHYSICS AND COMPUTER SCIENCE IT IS COMMON TO MODEL SYSTEMS USING STATE SPACES. PATHS IN SUCH SPACES REPRESENT HOW THE SYSTEM EVOLVES THROUGH TIME.

BUT MANY SYSTEMS ARE IRREVERSIBLE

PATHS SHOULD POINT FORWARDS IN TIME



IT DOESN'T MAKE SENSE TO TRAVERSE THESE "TIME-LIKE" PATHS BACKWARDS SO THEY MAY NOT HAVE INVERSES.

SUCH CONSIDERATIONS LEAD US TO

DIRECTED SPACES.

THESE INCORPORATE SOME KIND OF ORDER, REPRESENTING NOTIONS SUCH AS CAUSALITY OR RESOURCE USE

ON APPLYING THE "PATHS CONSTRUCTION" TO A DIRECTED SPACE, RESTRICTING OURSELVES TO CONSIDER DIRECTED PATHS ONLY, WE OBTAIN AN ∞-CATEGORY.

THE FUNDAMENTAL ∞-CATEGORY $\Pi(X)$

CLAIM WE MAY FRUITFULLY STUDY $\Pi(X)$ USING GENERALISATIONS OF STANDARD CATEGORICAL TOOLS....

BUT HOW?

WHEN WE START LOOKING...

... EXAMPLES OF ∞-CATEGORIES ARE ABUNDANT IN "NATURE".

- DERIVED ALGEBRAIC GEOMETRY
- FORMAL FOUNDATIONS AND TYPE THEORY
- ALGEBRAIC TOPOLOGY
- QUANTUM FIELD THEORY + QUANTUM COMPUTATION
- CATEGORY THEORY ITSELF

. . . AND MUCH MORE.

SLOGAN

∞-CATEGORIES $\equiv$ CATEGORIES + EQUALITY AS HOMOTOPY COHERENT STRUCTURE

THEY MAY BE UBIQUITOUS BUT YOU HAVEN'T ACTUALLY DEFINED WHAT THEY ARE!*

* AND I WON'T!!

AN INTUITIVE MODEL

$(\infty,1)$-CATEGORY $\approx$ CATEGORY WEAKLY ENRICHED IN SPACES

A

SET OF OBJECTS

$ob(A)$

HOM SPACES

$A(a,b)$

CONTINUOUS COMPOSITION OPERATIONS

$$A(b,c) \times A(a,b) \xrightarrow{\circ} A(a,c)$$

ASSOCIATIVITY HOMOTOPIES

$$\begin{tikzcd}
 & K(c,d) \times K(a,c) \arrow[dr, "\circ"] & \\
K(c,d) \times K(b,c) \times K(a,b) \arrow[ur, "K(c,d)\times\circ"] \arrow[dr, "\circ\times K(a,b)"'] & \simeq & K(a,d) \\
 & K(b,d) \times K(a,b) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

$\equiv$ CONTINUOUS FAMILY OF PATHS

$$h\circ(g\circ f) \sim (h\circ g)\circ f$$

IN $K(a,d)$

MACLANE PENTAGON "HIGHER" HOMOTOPY

$$
\begin{tikzcd}[column sep=small]
 & (k \circ h) \circ (g \circ f) \arrow[dl, phantom, "\sim"] \arrow[dr, phantom, "\sim"] & \\
k \circ (h \circ (g \circ f)) \arrow[d, phantom, "\wr"] & \Rrightarrow & ((k \circ h) \circ g) \circ f \arrow[d, phantom, "\wr"] \\
k \circ ((h \circ g) \circ f) & \sim & (k \circ (h \circ g)) \circ f
\end{tikzcd}
$$

A HOMOTOPY OF HOMOTOPIES

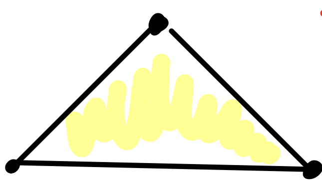
+ EVEN HIGHER COHERENCE HOMOTOPIES AT ALL LEVELS ABOVE THIS.... UGGHHH!

FORMALISING ....

... WE CAN MAKE PROGRESS ON FORMALISING THIS IDEA WITH TWO KEY OBSERVATIONS:

1) GIVEN COMPOSABLE ARROWS $g, f \in A$ IT MAY NOT MAKE GOOD SENSE TO SPEAK OF THE COMPOSITE $g \circ f$ BECAUSE ANY ARROW HOMOTOPIC TO IT HAS EQUAL RIGHT TO THAT CROWN.

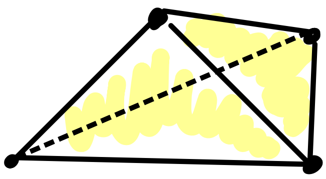
2) HIGHER ASSOCIATIVITY LAWS ARISE FROM THE ALGEBRAIC STRUCTURE OF THE HIGHER SIMPLICES.

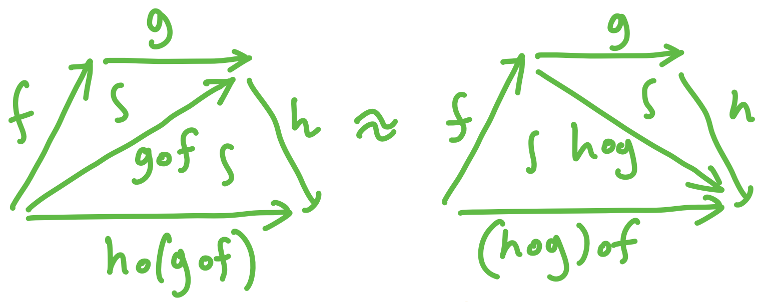


2-SIMPLEX

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "s"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & {} & \bullet \end{tikzcd}$$

3-SIMPLEX

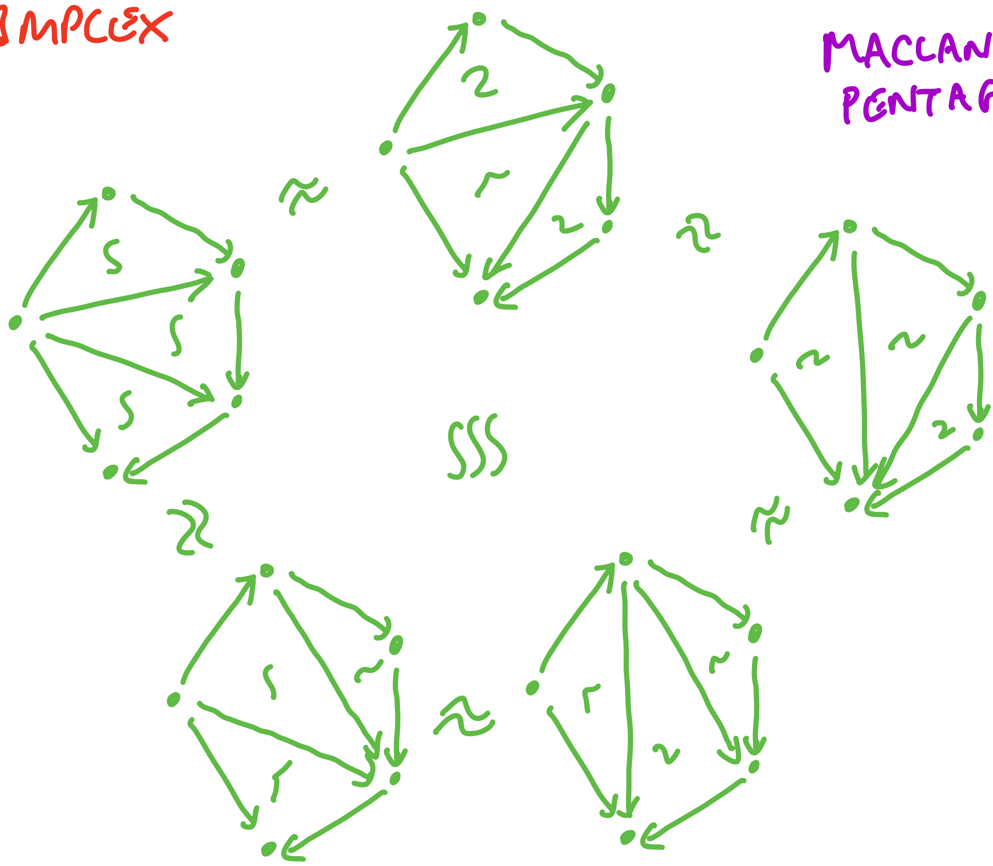




FOR THIS TO BE A WELL DEFINED 3-SIMPLEX THESE MUST BE CHOSEN TO BE EQUAL

4-SIMPLEX

MACLANE PENTAGON



VARIOUS METHODS HAVE BEEN EXPLORED TO DIRECTLY FORMALISE OUR INTUITIVE MODEL USING THESE IDEAS.

THESE HAVE GIVEN RISE TO A NUMBER OF FORMAL MODELS OF THE ∞-CATEGORY ZEITGEIST, FOR EXAMPLE:

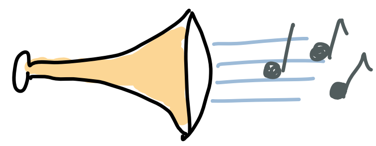
- (COMPLETE) SEGAL SPACES
- SEGAL CATEGORIES
- TAMSAMANI CATEGORIES

WE'LL LOOK, VERY BRIEFLY, AT A SLIGHTLY DIFFERENT KIND OF MODEL:

QUASI-CATEGORIES

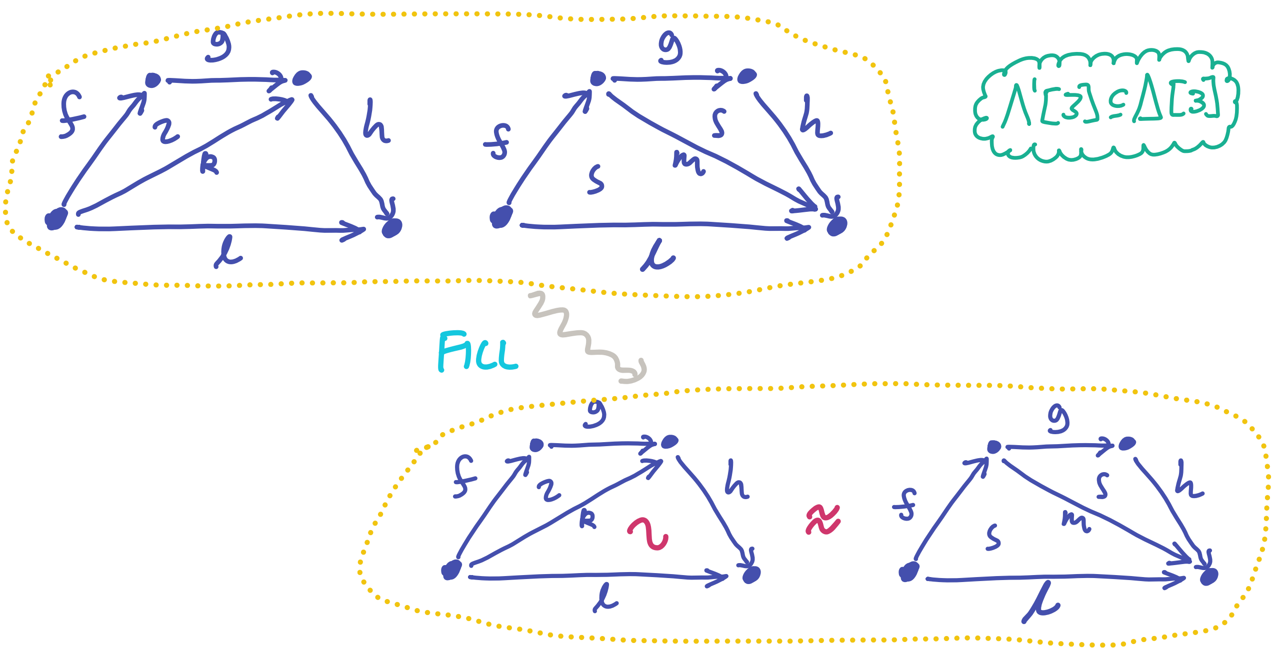
A QUASI-CATEGORY A IS A SIMPLICIAL SET WHICH ADMITS FILLERS FOR INNER HORNS.

A COMBINATORIAL SIMPLICIAL SPACE



$\Lambda^1[2] \subseteq \Delta[2]$

$\Lambda^1[3] \subseteq \Delta[3]$



OK...

... SO NOW WE HAVE SOME (ARGUABLY TOO MANY) FORMAL STRUCTURES THAT CLAIM TO MODEL OUR INTUITION ABOUT ∞-CATEGORIES ...

... BUT YOU PROMISED WE'D DO SOME CATEGORY THEORY WITH THEM ...

... HOW DOES THAT WORK?

LIMITS AND COLIMITS?

COMMA ∞-CATS

ADJUNCTIONS?

MONADS AND MONADICITY?

PRESHEAF CATEGORIES?

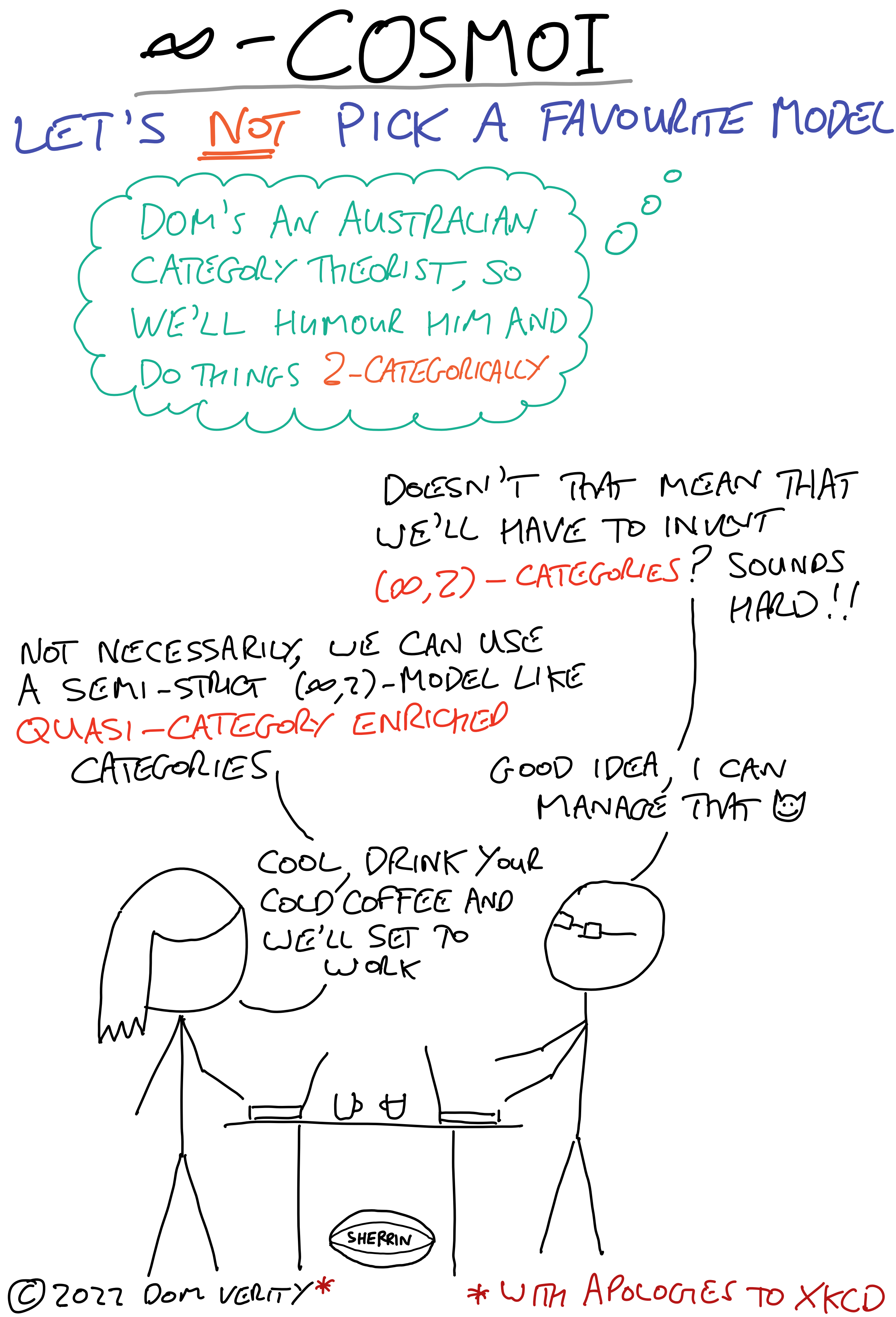
GROTHENDIECK FIBRATIONS?

TOPOSES?

POINTWISE KAN EXTENSIONS

YONEDA?

... AND WHAT MODEL SHOULD WE USE?


∞-COSMOI
LET'S NOT PICK A FAVOURITE MODEL
DOM'S AN AUSTRALIAN CATEGORY THEORIST, SO WE'LL HUMOUR HIM AND DO THINGS 2-CATEGORICALLY
DOESN'T THAT MEAN THAT WE'LL HAVE TO INVENT (∞,2)-CATEGORIES? SOUNDS HARD!!
NOT NECESSARILY, WE CAN USE A SEMI-STRICT (∞,2)-MODEL LIKE QUASI-CATEGORY ENRICHED CATEGORIES
GOOD IDEA, I CAN MANAGE THAT
COOL, DRINK YOUR COLD COFFEE AND WE'LL SET TO WORK
SHERRIN
©2022 DOM VERITY*
* WITH APOLOGIES TO XKCD

TECHNICALLY SPEAKING

∞-COSMOS $\mathcal{K}$ $\equiv$ CATEGORY OF FIBRANT OBJECT ENRICHED IN THE JOYAL MODEL STRUCTURE ON SIMPLICIAL SETS

THESE HAVE

1-CELLS

2-CELLS

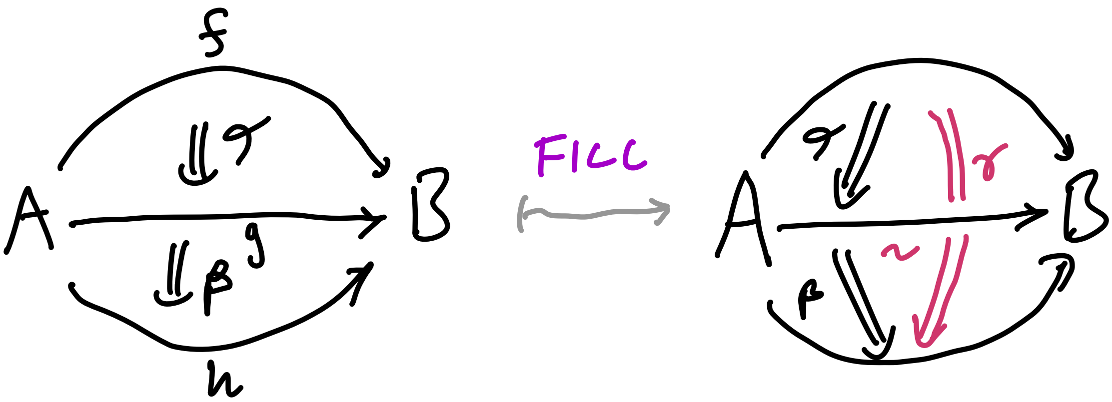
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$$

0-CELLS

+ SIMPLICIAL HIGHER HOMOTOPIES RELATING 2-CELLS

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "k\circ g"'] \arrow[r, phantom, "\Downarrow\gamma\circ\beta"] & C \end{tikzcd}$$

HORIZONTAL COMPOSITION IS A STRICTLY ASSOCIATIVE OPERATION



VERTICAL COMPOSITION GIVEN BY FILLING INNER HORNS.

THIS IS STILL A LITTLE MORE COMPLICATED THAN WE MIGHT HOPE FOR

FOR A SURPRISINGLY LARGE RANGE OF PURPOSES IT SUFFICES TO WORK IN A QUOTIENT OF $\mathcal{K}$ CALLED ITS HOMOTOPY 2-CATEGORY

HOMOTOPY 2-CATEGORIES

THE HOMOTOPY 2-CATEGORY $h_*\mathcal{K}$ OF AN ∞-COSMOS $\mathcal{K}$ HAS

- 0-CELLS AND 1-CELLS THOSE OF $\mathcal{K}$
- 2-CELLS HOMOTOPY CLASSES OF PARALLEL 2-CELLS IN $\mathcal{K}$
- HORIZONTAL COMPOSITION AS IN $\mathcal{K}$
- VERTICAL COMPOSITION USE FILLERS TO COMPOSE REPRESENTATIVES OF HOMOTOPY CLASSES.

THIS IS A GOOD, OLD FASHIONED, STRICT 2-CATEGORY

QUESTION WHAT DO WE GET IF WE STUDY STANDARD 2-CATEGORICAL STRUCTURES IN THE HOMOTOPY 2-CATEGORY $h_*\mathcal{K}$?

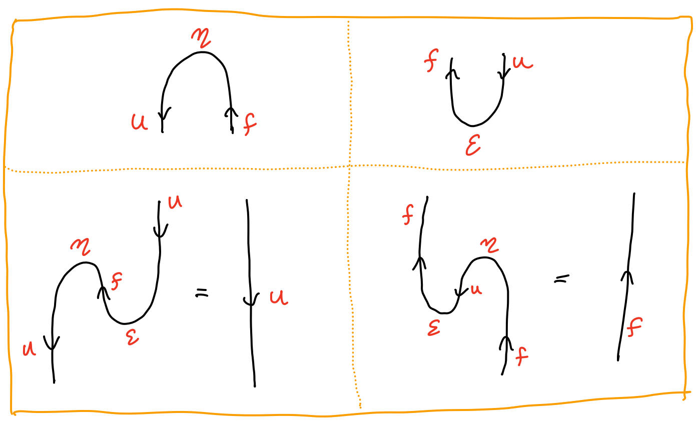
CASE STUDY: ADJUNCTIONS

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

$$[\eta] : id_B \Rightarrow uf \qquad\qquad [\varepsilon]f \cdot f[\eta] = id_f$$

$$[\varepsilon] : fu \Rightarrow id_A \quad + \quad u[\varepsilon] \cdot [\eta]u = id_u$$

TRIANGLE IDENTITIES.



IT TURNS OUT THAT WHEN INTERPRETED IN VARIOUS ∞-MODELS THIS ADJUNCTION NOTION COINCIDES WITH ALL OTHER SUCH NOTIONS IN THE LITERATURE

IN QUASI-CATEGORIES, FOR EXAMPLE, SUCH ADJUNCTIONS ARE CHARACTERISED BY A HIGHER UNIVERSAL PROPERTY

EXPRESSED AS A KIND OF "OUTER" HORN FILLING PROPERTY

$$\begin{tikzcd} b \arrow[r, "\eta_b"] \arrow[dr, "\alpha"'] & ufb \\ & ua \end{tikzcd} \quad B$$

$$\begin{tikzcd} b \arrow[r, "\eta_b"] \arrow[dr, "\alpha"'] & ufb \arrow[d, "f\hat{\alpha}"] \arrow[dl, phantom, "\sim"] \\ & ua \end{tikzcd} \quad B \qquad \begin{tikzcd} fb \arrow[d, "\hat{\alpha}"] \\ a \end{tikzcd} \quad A$$

+ HIGHER VERSIONS OF THIS PROPERTY AT ALL HIGHER DIMENSIONS.

THIS IS A LITTLE SURPRISING, WE GET HIGHER UNIVERSAL PROPERTIES FROM A 2-DIMENSIONAL NOTION INTERPRETED IN h_*K

AS AN ADDED BENEFIT, WE FIND THAT ANY 2-CATEGORICAL PROPERTY OF ADJUNCTIONS HOLDS FOR ADJUNCTIONS IN ANY ∞-COSMOS

THE 2-CATEGORICAL PROOFS OF THESE ARE OFTEN MUCH SIMPLER THAN THE ∞-CATEGORICAL ONES TO BE FOUND IN THE LITERATURE ON SPECIFIC MODELS

HOW MUCH...

...CAN WE CAPTURE IN THIS WAY

LIMITS AND COLIMITS?

COMMA ∞-CATS

ADJUNCTIONS?

MONADS AND MONADICITY?

PRESHEAF CATEGORIES?

GROTHENDIECK FIBRATIONS?

TOPOSES?

POINTWISE KAN EXTENSIONS

YONEDA?

NEEDS A LITTLE EXTRA 2-CATEGORICAL STRUCTURE

THE MONAD STORY CANNOT ENTIRELY BE UNDERTAKEN IN $h_*\mathcal{K}$, IT NEEDS A HOMOTOPY COHERENCE RESULT FOR ADJUNCTIONS.

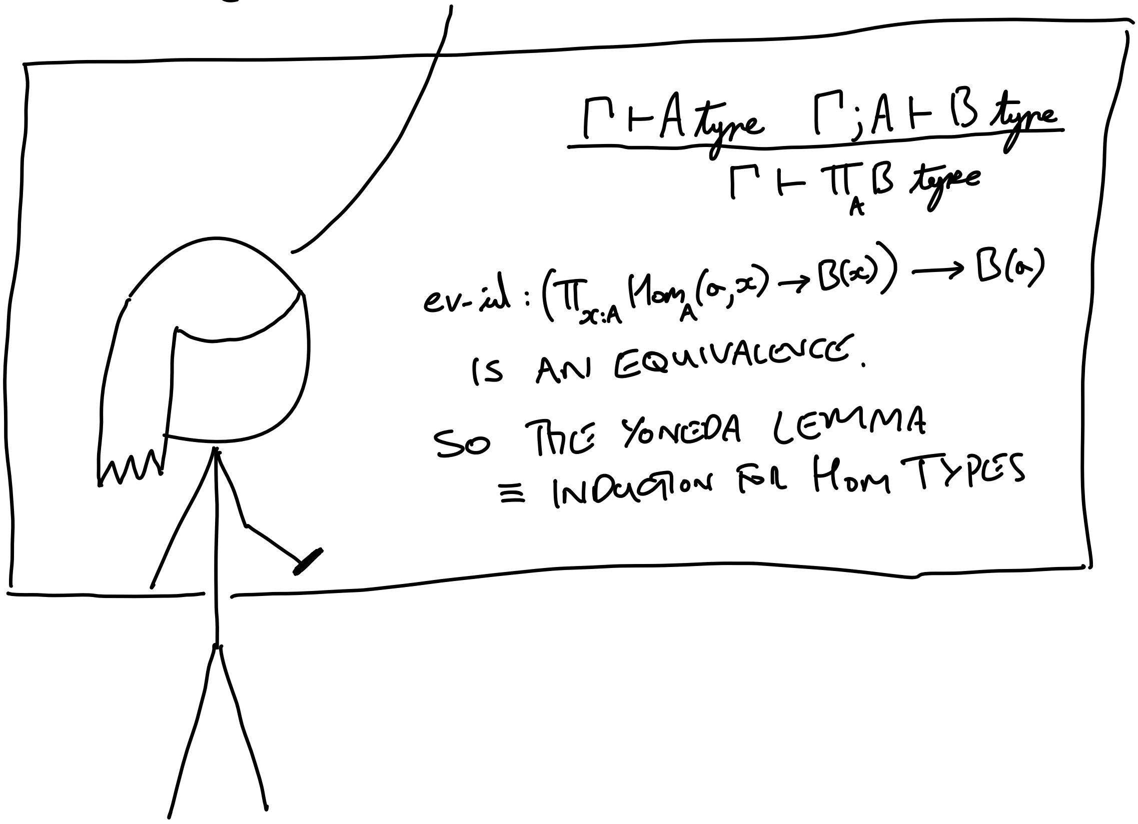
APPENDIX

∞-CATEGORIES IN HOMOTOPY TYPE THEORY (HoTT)

HoTT IS DESIGNED SO THAT EQUALITIES NATIVELY BEHAVE LIKE HOMOTOPIES

THEY ARE STRUCTURES, NOT RELATIONS, AND THEY SUPPORT HIGHER STRUCTURES

TERMS IN THESE STRUCTURES COLLECT INTO AN ∞-GROUPOID



IN RIEHL AND SHULMAN'S DIRECTED TYPE THEORY ALL TYPES A SUPPORT A FAMILY OF HOM TYPES

$$\mathrm{Hom}_A(x,y) \text{ OVER } x,y : A$$

THAT TYPE IS SAID TO BE AN ∞-CATEGORY IF

- EVERY PAIR OF ARROWS $f : \mathrm{Hom}_A(x,y)$ AND $g : \mathrm{Hom}_A(y,z)$ HAS A UNIQUE COMPOSITE DEFINING A TERM $g \circ f : \mathrm{Hom}_A(x,z)$
- PATHS IN A ARE EQUIVALENT TO ISOMORPHISMS

BY PATHS WE MEAN TERMS IN AN IDENTITY TYPE $\mathrm{Id}_A(x,y)$ FOR $x,y : A$

THINK OF THIS AS THE SPACE OF PATHS (HOMOTOPIES) FROM x TO y.

THE UNIQUENESS OF THIS COMPOSITE IS A LITTLE SURPRISING!!

BUT, HOMOTOPY THEORY IS BUILT INTO THE VERY NOTION OF EQUALITY IN THE UNDERLYING TYPE THEORY.

SO UNIQUENESS DECODES TO GIVE THE ENTIRELY PALATABLE STATEMENT THAT:

"THE ∞-GROUPOID (SPACE) OF COMPOSITES IS CONTRACTIBLE"